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Shape reconstruction of an inverse boundary value problem of two-dimensional Navier–Stokes equations

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SUMMARY

This paper is concerned with the problem of the shape reconstruction of two-dimensional flows governed by the Navier–Stokes equations. Our objective is to derive a regularized Gauss–Newton method using the corresponding operator equation in which the unknown is the geometric domain. The theoretical foundation for the Gauss–Newton method is given by establishing the differentiability of the initial boundary value problem with respect to the boundary curve in the sense of a domain derivative. The numerical examples show that our theory is useful for practical purpose and the proposed algorithm is feasible. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we are interested in the identification of an obstacle immersed in a fluid driven by the two-dimensional Navier–Stokes equations. This problem arises in aerospace, automotive, hydraulic, ocean, structural and wind engineering. Example applications include aerodynamic design of automotive vehicles, train, low-speed aircraft and hydrodynamic design for ship hulls, turbomachinery and offshore structures.

Early works concerning with the domain derivative have been addressed in [1, 2]. Kirsch and Hettlich solved the inverse obstacle scattering problem for sound soft and sound hard obstacles

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by the domain derivative method, and in [3], Hettlich considers an inverse conductive scattering problem using the domain derivative. In [4, 5], the three authors applied the domain derivative to deal with the inverse boundary problem for the time-dependent heat equation in the case of perfectly conducting and insulating inclusions. In [6], we solved a shape reconstruction problem for heat conduction with mixed condition. Moreover, we derived the expressions of domain derivative for the inverse Stokes problem and investigate the numerical simulation by the regularized Gauss–Newton iterative method in [7].

In this paper, we extend the concept of the domain derivative to two-dimensional flows governed by Navier–Stokes equations, and present the efficient numerical algorithm for the two-dimensional realizations of the shape reconstruction problems.

The divergence-free condition coming from the fact that the fluid has a homogeneous density and evolves as an incompressible flow, and it is very difficult to impose on the mathematical and numerical point of view. We use Piola transformation to bypass the divergence-free condition for the flow.

This paper is organized into three parts. In the remainder of the section we establish the notation that will be used throughout the work. Section 2 is devoted to introduce Piola transformation for divergence-free condition and we establish the differentiability of the solution with respect to the boundary. This will serve as the theoretical foundation of the Newton method for the approximation solution considered in the third part of the paper. The third section describes regularized Newton schemes applied to the numerical inverse problem. The results of several numerical experiments show that the iterative algorithm gives good reconstruction and our theoretical work is correct.

Throughout the paper we will use the standard notation for Sobolev spaces (see [8]). Specially $H^r(\Omega)$, where r is an integer greater than zero, will denote the Sobolev space of real-valued functions with square integrable derivatives of order up to r equipped with the usual norm which we denote $\|\cdot\|_r$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$, and the standard L^2 inner product by $(\cdot,\cdot)$. Also $\mathbf{H^r(\Omega)}$ will denote the space of vector-valued functions each of whose n components belong to $H^r(\Omega)$. We introduce the space

$$\mathbf{H_0^1}(\Omega) := \{\mathbf{v} \in \mathbf{H^1}(\Omega), \mathbf{v}|_{\partial\Omega} = 0\}$$

$$\mathbf{H^1}(\mathrm{div}, \Omega) := \{\mathbf{v} \in \mathbf{H^1}(\Omega), \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\}$$

and

$$\mathbf{H_0^1}(\mathrm{div}, \Omega) := \{\mathbf{v} \in \mathbf{H^1}(\Omega), \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}|_{\partial\Omega} = 0\}$$

2. DOMAIN DERIVATIVE

We assume that Ω_1 and Ω_2 are two simply connected bounded domains of class C^2 in $\mathbb{R}^N$ ($N=2$ or 3), such that $\bar{\Omega}_2 \subset \Omega_1$. The boundaries of Ω_1 and Ω_2 are denoted by Γ_1 and Γ_2, respectively. Further, we denote $\mathbf{\Omega} := \Omega_1 \setminus \bar{\Omega}_2$, and let $\mathbf{f} \in \mathbf{L}^2(\Omega)$ be a given vector function in $\mathbf{\Omega}$. We seek a vector function $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ representing the velocity of the fluid, and a scalar function

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p representing the pressure, which are defined in Ω and satisfy the following equations and the boundary conditions (v is the coefficient of kinematic viscosity):

$$\begin{aligned} -v\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \Gamma_1 \\ \mathbf{u} &= 0 \quad \text{on } \Gamma_2 \end{aligned} \tag{1}$$

If $\mathbf{f}, \mathbf{u}$ and p are smooth functions satisfying (1) then, taking the scalar product of (1) with a function $\mathbf{v}\in\mathbf{H}_0^1(\operatorname{div},\Omega)$, we obtain

$$\begin{aligned} &\text{seek } \mathbf{u}\in\mathbf{H}_0^1(\operatorname{div},\Omega) \text{ such that} \\ &a(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\operatorname{div},\Omega) \end{aligned} \tag{2}$$

where

$$a(\mathbf{u},\mathbf{v})=v\sum_{i,j=1}^{n}\int_\Omega (D_i u_j)(D_i v_j)\,\mathrm{d}x$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=\sum_{i,j=1}^{n}\int_\Omega u_i(D_i v_j)w_j\,\mathrm{d}x$$

Taking $\mathbf{v}=\mathbf{u}$ in (2), we derive

$$\|\mathbf{u}\|_1\leqslant v^{-1}\|\mathbf{f}\|_0 \tag{3}$$

Continuity of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ can be demonstrated. And under the assumption,

$$\|b\|v^{-2}\|\mathbf{f}\|_0<1$$

the condition guarantees the existence and uniqueness of a solution $\mathbf{u}$ (see [9, 10]).

Let a perturbation of the interior boundary Γ_2 be specified by

$$\Gamma_2^h=\{\mathbf{x}+\mathbf{h}(\mathbf{x}), \mathbf{x}\in\Gamma_2\}$$

which is a C^2 boundary of a perturbed domain Ω_h, if the vector field $\mathbf{h}\in C^2(\Gamma_2)$ is sufficiently small. We choose an extension of $\mathbf{h}\in C^2(\Omega)$ with $\|\mathbf{h}\|_{C^2(\Omega)}\leqslant c\|\mathbf{h}\|_{C^2(\Gamma_2)}, c>0$, which vanishes in the exterior of a neighbourhood of Γ_2, and define the diffeomorphism $\varphi(\mathbf{x})=\mathbf{x}+\mathbf{h}(\mathbf{x})$ in Ω. If the inverse function of φ is denoted by ψ, J_φ, J_ψ and J_h are Jacobian matrices.

Lemma 2.1 (*Delfour and Zolésio* [*11*])
The Piola transformation

$$\mathbb{P}:\mathbf{w}\to(\det(J_\varphi)^{-1}J_\varphi\tilde{\mathbf{w}})\circ\psi$$

is an isomorphism, where $\mathbf{w}\in\mathbf{H}_0^1(\operatorname{div},\Omega_h)$ and $\tilde{\mathbf{w}}\in\mathbf{H}_0^1(\operatorname{div},\Omega)$.

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It still satisfies the condition of divergence free by the transformation, i.e.

$$\begin{aligned}
&\int_{\Omega_h} \operatorname{div} \mathbf{w} \cdot q_h \, \mathrm{d}x_h \\
&\quad = -\int_{\Omega_h} \mathbf{w} \cdot \nabla q_h \, \mathrm{d}x_h + \int_{\partial\Omega_h} \mathbf{w} \cdot q_h \cdot \mathbf{n} \, \mathrm{d}s_h \\
&\quad = -\int_{\Omega_h} (\det(J_\varphi)^{-1} J_\varphi \tilde{\mathbf{w}} \circ \psi) \cdot \nabla (q \circ \psi) \, \mathrm{d}x_h \\
&\quad = -\int_{\Omega_h} (\det(J_\varphi)^{-1} J_\varphi \tilde{\mathbf{w}} \circ \psi) \cdot (\nabla q \, J_\varphi^{-1} \circ \psi) \, \mathrm{d}x_h \\
&\quad = -\int_{\Omega} \tilde{\mathbf{w}} \cdot \nabla q \, \mathrm{d}x = \int_{\Omega} \operatorname{div} \tilde{\mathbf{w}} \cdot q \, \mathrm{d}x
\end{aligned}$$

Let $\mathbf{u}_h \in \mathbf{H}_\mathbf{0}^\mathbf{1}(\operatorname{div}, \Omega_h)$ be the solution of corresponding boundary value problem,

$$v \int_{\Omega_h} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h \, \mathrm{d}x_h + \int_{\Omega_h} (\mathbf{u}_h \cdot \nabla) \mathbf{u}_h \cdot \mathbf{v}_h \, \mathrm{d}x_h = \int_{\Omega_h} \mathbf{f}_h \cdot \mathbf{v}_h \, \mathrm{d}x_h \quad \forall \mathbf{v}_h \in \mathbf{H}_\mathbf{0}^\mathbf{1}(\operatorname{div}, \Omega_h) \tag{4}$$

Changing the variables by the Piola transformation leads to

$$v \int_{\Omega} \nabla (B\tilde{\mathbf{u}}) \cdot \nabla (B\mathbf{v}) A \, \mathrm{d}x + \int_{\Omega} (B\tilde{\mathbf{u}} \cdot \nabla) B\tilde{\mathbf{u}} \cdot \mathbf{v} \, \mathrm{d}x = \int_{\Omega} \tilde{\mathbf{f}} \cdot \mathbf{v} J_\varphi \, \mathrm{d}x \quad \forall \mathbf{v} \in \mathbf{H}_\mathbf{0}^\mathbf{1}(\operatorname{div}, \Omega) \tag{5}$$

where the notations $B = \det(J_\varphi)^{-1} J_\varphi$, $A = J_\varphi^{-1} (J_\varphi^{-1})^{\mathrm{T}} \det(J_\varphi)$ and $\tilde{\mathbf{f}} = \mathbf{f}_h \circ \psi$.

From $J_\varphi = I + J_h$ and $J_\psi = J_\varphi^{-1} \circ \psi = I - J_h + O(\|\mathbf{h}\|^2_{C^2(\Omega)})$, the following estimates hold:

$$\| J_\varphi^{-1} (J_\varphi^{-1})^{\mathrm{T}} \det(J_\varphi) - I + J_h + J_h^{\mathrm{T}} - \operatorname{div} \mathbf{h} \cdot I \|_\infty = O(\|\mathbf{h}\|^2_{C^2(\Omega)}) \tag{6}$$

$$\|\tilde{\mathbf{f}} \cdot J_\varphi - \mathbf{f} - \mathbf{f} \cdot \operatorname{div} \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{f}\|_\infty = O(\|\mathbf{h}\|^2_{C^2(\Omega)}) \tag{7}$$

and

$$\| J_\varphi^{-1} \det(J_\varphi) - I + J_h - \operatorname{div} \mathbf{h} \cdot I \|_\infty = O(\|\mathbf{h}\|^2_{C^2(\Omega)}) \tag{8}$$

At first, we prove the norm estimate (6). From $J_\varphi = I + J_h$, we can obtain

$$\begin{aligned}
&J_\varphi^{-1} (J_\varphi^{-1})^{\mathrm{T}} \det(J_\varphi) - I + J_h + J_h^{\mathrm{T}} - \operatorname{div} \mathbf{h} \cdot I \\
&\quad = (I - J_h + O(\|\mathbf{h}\|^2_{C^2(\Omega)}))(I - J_h^{\mathrm{T}} + O(\|\mathbf{h}\|^2_{C^2(\Omega)}))(1 + \operatorname{div} \mathbf{h} + O(\|\mathbf{h}\|^2_{C^2(\Omega)})) \\
&\qquad - I + J_h + J_h^{\mathrm{T}} - \operatorname{div} \mathbf{h} \cdot I \\
&\quad = (I - J_h - J_h^{\mathrm{T}} + O(\|\mathbf{h}\|^2_{C^2(\Omega)}))(1 + \operatorname{div} \mathbf{h} + O(\|\mathbf{h}\|^2_{C^2(\Omega)})) - I + J_h + J_h^{\mathrm{T}} - \operatorname{div} \mathbf{h} \cdot I
\end{aligned}$$

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$$=I-J_h-J_h^{\mathrm{T}}+\operatorname{div}\mathbf{h}\cdot I+O(\|\mathbf{h}\|^2_{C^2(\Omega)})-I+J_h+J_h^{\mathrm{T}}-\operatorname{div}\mathbf{h}\cdot I$$

$$=O(\|\mathbf{h}\|^2_{C^2(\Omega)})$$

Similarly, we can get the norm approximations (7) and (8).

Lemma 2.2 (*Hettlich* [*2, 3*])
If $u_i, v_i \in H_0^1(\Omega), i=1,\ldots,N$, then the following identity holds:

$$\nabla u_i(J_h+J_h^{\mathrm{T}}-\operatorname{div}\mathbf{h}\cdot I)\nabla v_i=\operatorname{div}[\mathbf{d}_i]-(\mathbf{h}\cdot\nabla u_i)\Delta v_i-(\mathbf{h}\cdot\nabla v_i)\Delta u_i \tag{9}$$

where $\mathbf{d}_i=(\mathbf{h}\cdot\nabla u_i)\nabla v_i+(\mathbf{h}\cdot\nabla v_i)\nabla u_i-(\nabla u_i\cdot\nabla v_i)\mathbf{h}$.

Lemma 2.3 (*Delfour and Zolésio* [*11*])
Let $w\in C^2(\Gamma)$ be a scalar function, and a vector field $\mathbf{v}\in C^1(\Gamma)^N$. The following decompositions hold:

$$\nabla w=\nabla_\tau w+\partial_{\mathbf{n}} w\mathbf{n} \tag{10}$$

$$\mathbf{v}=(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+v_\tau,\quad v_\tau=\mathbf{n}\wedge(\mathbf{v}\wedge\mathbf{n}) \tag{11}$$

Theorem 2.1
Let $\mathbf{f}$ belong to $\mathbf{L}^2(\Omega)$, $\mathbf{u}\in\mathbf{H}_0^1(\operatorname{div},\Omega)$ denote the solution of (1), and $\tilde{\mathbf{u}}$ is defined in (5). Then $\mathbf{u}$ is differentiable at Γ_2 in the sense that there exists $\mathbf{u}^*$ depending on $\mathbf{h}$, such that

$$\lim_{\mathbf{h}\to 0}\frac{1}{\|\mathbf{h}\|_{C^2}}\|\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*\|_1=0 \tag{12}$$

Furthermore $\mathbf{u}^*=\mathbf{u}'+(\mathbf{h}\cdot\nabla)\mathbf{u}$, where the domain derivative $\mathbf{u}'$ is defined by the solution of the boundary value problem

$$\begin{aligned}-\nu\Delta\mathbf{u}'+(\mathbf{u}\cdot\nabla)\mathbf{u}'+(\mathbf{u}'\cdot\nabla)\mathbf{u}+\nabla p'&=0\quad\text{in }\Omega\\ \operatorname{div}\mathbf{u}'&=0\quad\text{in }\Omega\\ \mathbf{u}'&=0\quad\text{on }\Gamma_1\\ \mathbf{u}'&=-\mathbf{h}_n\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\quad\text{on }\Gamma_2\end{aligned} \tag{13}$$

where $\mathbf{h}_n=\mathbf{h}\cdot\mathbf{n}$ is the normal component of the vector field $\mathbf{h}$.

Proof
First of all, we establish continuous dependence of the solution $\mathbf{u}$ on variations of the boundary Γ_2. Then we will prove the differentiability of the solution with respect to the boundary Γ_2, and deduce the domain derivative of Navier–Stokes equations. □

We consider the difference $\tilde{\mathbf{u}}-\mathbf{u}$, and the variational equation yields

$$a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})+b(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})$$

$$=a(\tilde{\mathbf{u}},\mathbf{v})-a(B\tilde{\mathbf{u}},\mathbf{v})+a(B\tilde{\mathbf{u}},\mathbf{v})-a(B\tilde{\mathbf{u}},B\mathbf{v})$$

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$$+a(B\tilde{\mathbf{u}}, B\mathbf{v}) - \nu\int_\Omega \nabla(B\tilde{\mathbf{u}})\cdot\nabla(B\mathbf{v})A\,\mathrm{d}x + \nu\int_\Omega \nabla(B\tilde{\mathbf{u}})\cdot\nabla(B\mathbf{v})A\,\mathrm{d}x$$

$$-a(\mathbf{u},\mathbf{v}) - b(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\tilde{\mathbf{u}},\tilde{\mathbf{u}},\mathbf{v}) - b(B\tilde{\mathbf{u}},\tilde{\mathbf{u}},\mathbf{v}) + b(B\tilde{\mathbf{u}},\tilde{\mathbf{u}},\mathbf{v})$$

$$-b(B\tilde{\mathbf{u}}, B\tilde{\mathbf{u}},\mathbf{v}) - b(\tilde{\mathbf{u}},\mathbf{u},\mathbf{v}) - b(\mathbf{u},\tilde{\mathbf{u}},\mathbf{v}) + b(B\tilde{\mathbf{u}}, B\tilde{\mathbf{u}},\mathbf{v}) + 2b(\mathbf{u},\mathbf{u},\mathbf{v})$$

From Equations (4) and (5), we obtain

$$\begin{aligned}
&a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v}) + b(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{u}}-\mathbf{u},v)\\
&\quad = a((I-B)\tilde{\mathbf{u}},\mathbf{v}) + a(B\tilde{\mathbf{u}},(I-B)\mathbf{v}) + \nu\int_\Omega \nabla(B\tilde{\mathbf{u}})\cdot\nabla(B\mathbf{v})(I-A)\,\mathrm{d}x\\
&\qquad + \int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v}J_\varphi\,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x + b((I-B)\tilde{\mathbf{u}},\tilde{\mathbf{u}},\mathbf{v})\\
&\qquad + b(B\tilde{\mathbf{u}},(I-B)\tilde{\mathbf{u}},\mathbf{v}) + b(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u}-\tilde{\mathbf{u}},\mathbf{v})
\end{aligned}$$

Let $\mathbf{v}=\tilde{\mathbf{u}}-\mathbf{u}$ in the last expression, and recall the inequality (3). The perturbation argument shows the continuity in the light of the approximation (6)–(8),

$$\|\tilde{\mathbf{u}}-\mathbf{u}\|_1 \to 0 \quad \text{as } \|\mathbf{h}\|_{C^2(\Omega)} \to 0$$

In order to show the differentiability, let $\mathbf{u}^* \in \mathbf{H}_\mathbf{0}^\mathbf{1}(\Omega)$ be the solution of

$$\begin{aligned}
&a(\mathbf{u}^*,\mathbf{v}) + b(\mathbf{u}^*,\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u}^*,\mathbf{v})\\
&\quad = \nu\int_\Omega \nabla\mathbf{u}(J_h + J_h^\mathrm{T} - \operatorname{div}\mathbf{h}\cdot I)\cdot\nabla\mathbf{v}\,\mathrm{d}x + \int_\Omega (\operatorname{div}\mathbf{h}\cdot\mathbf{f} + (\mathbf{h}\cdot\nabla)\mathbf{f})\mathbf{v}\,\mathrm{d}x\\
&\qquad + \int_\Omega [((\mathbf{h}\cdot\nabla)\mathbf{u} - \operatorname{div}\mathbf{h}\cdot I)\cdot\nabla]\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x
\end{aligned} \tag{14}$$

for all $\mathbf{v}\in\mathbf{H}_\mathbf{0}^\mathbf{1}(\operatorname{div},\Omega)$.

According to the properties of forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$, the following expression holds:

$$\begin{aligned}
&a(\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v}) + b(\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v})\\
&\quad = a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v}) - a(\mathbf{u}^*,\mathbf{v}) + b(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})\\
&\qquad + b(\mathbf{u},\mathbf{u}^*,\mathbf{v}) + b(\mathbf{u}^*,\mathbf{u},\mathbf{v}) - b(\tilde{\mathbf{u}},\mathbf{u}^*,\mathbf{v}) - b(\mathbf{u}^*,\tilde{\mathbf{u}},\mathbf{v}) + b(\mathbf{u}^*,\mathbf{u}^*,\mathbf{v})\\
&\quad = a(\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v}) - a(\mathbf{u}^*,\mathbf{v}) + b(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{u}}-\mathbf{u},\mathbf{v})\\
&\qquad + b(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{u}^*,\mathbf{v}) - b(\mathbf{u}^*,\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v})
\end{aligned}$$

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Considering $\mathbf{u}^*$ is the solution of (14), and we derive

$$
\begin{aligned}
&a(\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v})+b(\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v})\\
&\quad=a((I-B)\tilde{\mathbf{u}},\mathbf{v})+a(B\tilde{\mathbf{u}},(I-B)\mathbf{v})+v\int_\Omega \nabla(B\tilde{\mathbf{u}})\cdot\nabla(B\mathbf{v})(I-A)\,\mathrm{d}x\\
&\qquad+\int_\Omega(\tilde{\mathbf{f}}\cdot J_\varphi-\mathbf{f})\cdot\mathbf{v}\,\mathrm{d}x+b((I-B)\tilde{\mathbf{u}},\tilde{\mathbf{u}},\mathbf{v})+b(B\tilde{\mathbf{u}},(I-B)\tilde{\mathbf{u}},\mathbf{v})\\
&\qquad-v\int_\Omega\nabla\mathbf{u}(J_h+J_h^{\mathrm{T}}-\operatorname{div}\mathbf{h}\cdot I)\cdot\nabla\mathbf{v}\,\mathrm{d}x-\int_\Omega(\operatorname{div}\mathbf{h}\cdot\mathbf{f}+(\mathbf{h}\cdot\nabla)\mathbf{f})\mathbf{v}\,\mathrm{d}x\\
&\qquad+b(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u}-\tilde{\mathbf{u}},\mathbf{v})+b(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{u}^*,\mathbf{v})-b(\mathbf{u}^*,\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*,\mathbf{v})
\end{aligned}
$$

Taking $\mathbf{v}=\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*$, and apply the norm estimates (6)–(8) again,

$$
\frac{1}{\|\mathbf{h}\|_{C^2}}\|\tilde{\mathbf{u}}-\mathbf{u}-\mathbf{u}^*\|_1\to 0\quad\text{as}\quad\|\mathbf{h}\|_{C^2}\to 0
$$

Next, we split $\mathbf{u}^*$ into $(\mathbf{h}\cdot\nabla)\mathbf{u}$ and $\mathbf{u}'$. In terms of the Lemmas 2.2 and 2.3 and the divergence formula, we obtain

$$
\begin{aligned}
\int_\Omega\operatorname{div}[\mathbf{d}]\,\mathrm{d}x&=\int_\Omega\operatorname{div}[((\mathbf{h}\cdot\nabla)\mathbf{u})\nabla\mathbf{v}+((\mathbf{h}\cdot\nabla)\mathbf{v})\nabla\mathbf{u}-(\nabla\mathbf{u}\cdot\nabla\mathbf{v})\mathbf{h}]\,\mathrm{d}x\\
&=\int_{\partial\Omega}[((\mathbf{h}\cdot\nabla)\mathbf{u})\nabla\mathbf{v}+((\mathbf{h}\cdot\nabla)\mathbf{v})\nabla\mathbf{u}-(\nabla\mathbf{u}\cdot\nabla\mathbf{v})\mathbf{h}]\cdot\mathbf{n}\,\mathrm{d}s\\
&=\int_{\partial\Omega}((\mathbf{h}\cdot\nabla)\mathbf{u})\nabla\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s
\end{aligned}
$$

Taking use of the Green formula, therefore,

$$
\begin{aligned}
a((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{v})&=-v\int_\Omega((\mathbf{h}\cdot\nabla)\mathbf{u})\Delta\mathbf{v}\,\mathrm{d}x+\int_{\partial\Omega}((\mathbf{h}\cdot\nabla)\mathbf{u})\nabla\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s\\
&=v\int_\Omega(\operatorname{div}[\mathbf{d}]-((\mathbf{h}\cdot\nabla)\mathbf{u})\Delta\mathbf{v}-((\mathbf{h}\cdot\nabla)\mathbf{v})\Delta\mathbf{u})\,\mathrm{d}x\\
&\quad-\int_\Omega((\mathbf{h}\cdot\nabla)\mathbf{v})(\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x\\
&=v\int_\Omega(\operatorname{div}[\mathbf{d}]-((\mathbf{h}\cdot\nabla)\mathbf{u})\Delta\mathbf{v}-((\mathbf{h}\cdot\nabla)\mathbf{v})\Delta\mathbf{u})\,\mathrm{d}x\\
&\quad+\int_\Omega\operatorname{div}\mathbf{h}\cdot\mathbf{v}(\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x+\int_\Omega\mathbf{v}((\mathbf{h}\cdot\nabla)(\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u}))\,\mathrm{d}x\\
&\quad-\int_{\partial\Omega}\mathbf{h}\cdot\mathbf{v}(\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}s
\end{aligned}
$$

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$$= \nu \int_\Omega (\operatorname{div}[\mathbf{d}] - ((\mathbf{h}\cdot\nabla)\mathbf{u})\Delta\mathbf{v} - ((\mathbf{h}\cdot\nabla)\mathbf{v})\Delta\mathbf{u})\,\mathrm{d}x$$

$$+ \int_\Omega \operatorname{div}\mathbf{h}\cdot\mathbf{v}(\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u})\,\mathrm{d}x + \int_\Omega \mathbf{v}((\mathbf{h}\cdot\nabla)(\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}))\,\mathrm{d}x$$

We add the trilinear forms to the last equality and utilize Lemma 2.2,

$$\begin{aligned}
&a((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{v}) + b(\mathbf{u},(\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{v}) + b((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{u},\mathbf{v})\\
&\quad = \nu\int_\Omega \nabla\mathbf{u}(J_h + J_h^{\mathrm{T}} - \operatorname{div}\mathbf{h}\cdot I)\cdot\nabla\mathbf{v}\,\mathrm{d}x + \int_\Omega (\operatorname{div}\mathbf{h}\cdot\mathbf{f} + (\mathbf{h}\cdot\nabla)\mathbf{f})\mathbf{v}\,\mathrm{d}x\\
&\qquad + \int_\Omega [(((\mathbf{h}\cdot\nabla)\mathbf{u})\cdot\nabla)\mathbf{u} - \operatorname{div}\mathbf{h}(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,\mathrm{d}x\\
&\quad = a(\mathbf{u}^*,\mathbf{v}) + b(\mathbf{u},\mathbf{u}^*,\mathbf{v}) + b(\mathbf{u}^*,\mathbf{u},\mathbf{v})
\end{aligned}$$

Thus, the following identity holds:

$$a(\mathbf{u}',\mathbf{v}) + b(\mathbf{u},\mathbf{u}',\mathbf{v}) + b(\mathbf{u}',\mathbf{u},\mathbf{v}) = 0$$

Since the perturbation is only on the boundary Γ_2, then

$$\mathbf{u}' = 0 \quad \text{on } \Gamma_1$$

It is known that $\mathbf{u}|_{\Gamma_2} = 0$ implies $\nabla_\tau \mathbf{u}|_{\Gamma_2} = 0$. Note that $\mathbf{u}^*$ vanishes on the boundary Γ_2,

$$\mathbf{u}' = \mathbf{u}^* - \mathbf{h}\cdot\nabla\mathbf{u} = -\left(\mathbf{h}_\tau\cdot\nabla_\tau\mathbf{u} + \mathbf{h}\cdot\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\mathbf{n}\right) = -\mathbf{h}_n\frac{\partial\mathbf{u}}{\partial\mathbf{n}}$$

Thus, $\mathbf{u}'$ satisfies the boundary value problem (13). The theorem is proved. □

3. NUMERICAL EXAMPLES

This section describes the essential step of an iterative algorithm for the inverse problem, which we formulate in two dimension. Newton method is based on the observation that the solution to the problem (1) defines an operator F on set X of admissible boundaries by

$$F(\Gamma_2) = P \tag{15}$$

where $X := \{\varphi \in C^2(\Gamma_2), 0 < \beta \leqslant \|\varphi\|_{C^2} \leqslant \gamma\}$, and φ is the parametrized form of the unknown interior boundary Γ_2. P is referred to as the measured data, based on the direct problem (1), P could be the measured normal press or the velocity on the boundary.

However, since the linearized version of (15) inherits the ill-posedness, the Newton iterations need to be regularized. This approach has the advantages that, in principle, it is conceptually simple and that it leads to highly accurate reconstructions. But, as disadvantages we note that the numerical implementation requires the forward solution of the problem (1) in each step of the Newton iteration and reasonable *a priori* information for the initial approximation.

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A numerical implementation requires a parametrization of the boundary. Here we use the parametric representations

$$\Gamma_k := \{x_k(t) = (x_{k,1}(t), x_{k,2}(t)), 0 \leqslant t < 2\pi\}, \quad k = 1, 2$$

where $x_k : \mathbb{R} \to \mathbb{R}^2$ is twice differentiable and 2π-periodic with $|x_k'(t)| > 0$ for all t, and assume that $\Gamma := \{x_2(t) : \sigma_1 \leqslant t \leqslant \sigma_2\}$. Further we assume that the orientation of the parametrization x_1 is clockwise and the parametrization x_2 is counter-clockwise. In addition, we assume that Γ_2 is starlike with respect to the origin, i.e.

$$x_\alpha(t) = r_\alpha(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad 0 \leqslant t \leqslant 2\pi \tag{16}$$

where

$$r_\alpha(t) = \alpha_0 + \sum_{j=1}^{M} (\alpha_j \cos jt + \alpha_{j+M} \sin jt)$$

with $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{2M})^{\mathrm{T}} \in \mathbb{R}^{2M+1}$ for some fixed number $M \in N$.

Let $U_M := \{\boldsymbol{\alpha} \in R^{2M+1} : \rho_1 \leqslant r_\alpha(t) \leqslant \rho_2, t \in [0, 2\pi]\}$ for some $0 < \rho_1 < \rho_2$. We can assign to each $\boldsymbol{\alpha} \in U_M$ the cost function $F(\Gamma_2)(x_i), i = 1, \ldots, Q$. In the following we fix M and Q. A simple application of Theorem 2.1 shows

Theorem 3.1

For $\boldsymbol{\alpha} \in U_M$ the mapping F is differentiable with $\partial F_i(\alpha)/\partial \alpha_j = \partial_n u_j{}'(x_i)$ for $i = 1, \ldots, Q$ and $j = 0, \ldots, 2M$. Here $u_j' \in H^1(\mathrm{div}, \Omega)$ is the solution of the boundary value problem

$$\begin{aligned} -v\Delta u_j' + (u_j \cdot \nabla) u_j' + (u_j' \cdot \nabla) u_j + \nabla p_j' &= 0 \quad \text{in } \Omega \\ \operatorname{div} u_j' &= 0 \quad \text{in } \Omega \\ u_j' &= 0 \quad \text{on } \Gamma_1 \\ u_j' &= -k \frac{\partial u_j}{\partial n_i} \quad \text{on } \Gamma_2 \end{aligned} \tag{17}$$

where

$$k = -\frac{r_\alpha(t)}{\sqrt{r_\alpha'(t)^2 + r_\alpha(t)^2}} \begin{cases} \cos jt, & j = 0, \ldots, M \\ \sin(j - M)t, & j = M+1, \ldots, 2M \end{cases}$$

for $t \in [0, 2\pi]$.

The numerical algorithm can be summarized as follows:

Step (1): Given an original boundary, we parametrize it to α^0.

Step (2): Solve the direct problem (1) by the finite element method.

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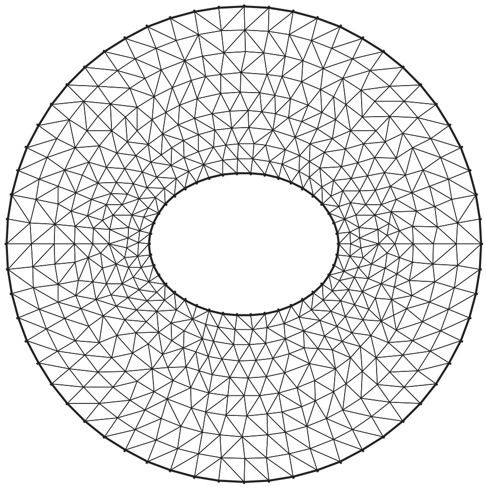

Figure 1. Initial mesh in case 1 with 362 nodes.

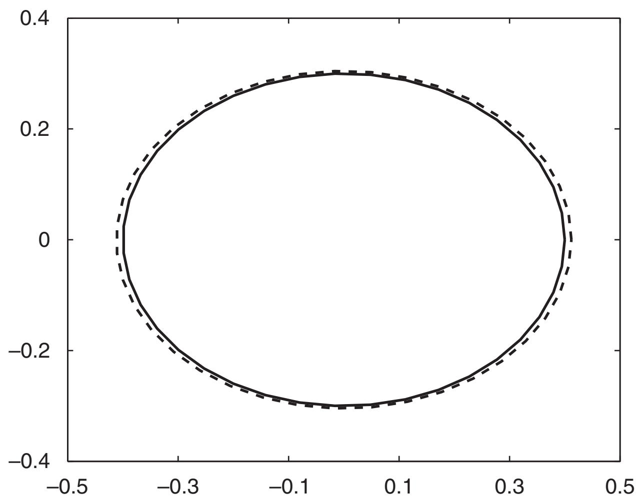

Figure 2. The numerical result of case 1 with $v = 0.02$.

Step (3): According to Theorem 3.1, for a given α^n, we calculate the Jacobian matrix $J(\alpha)$,

$$J(\alpha^{n+1}) = \begin{bmatrix} \dfrac{\partial F_1(\alpha^n)}{\partial \alpha_0^n} & \dfrac{\partial F_1(\alpha^n)}{\partial \alpha_1^n} & \cdots & \dfrac{\partial F_1(\alpha^n)}{\partial \alpha_{2M}^n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_Q(\alpha^n)}{\partial \alpha_0^n} & \dfrac{\partial F_Q(\alpha^n)}{\partial \alpha_1^n} & \cdots & \dfrac{\partial F_Q(\alpha^n)}{\partial \alpha_{2M}^n} \end{bmatrix}$$

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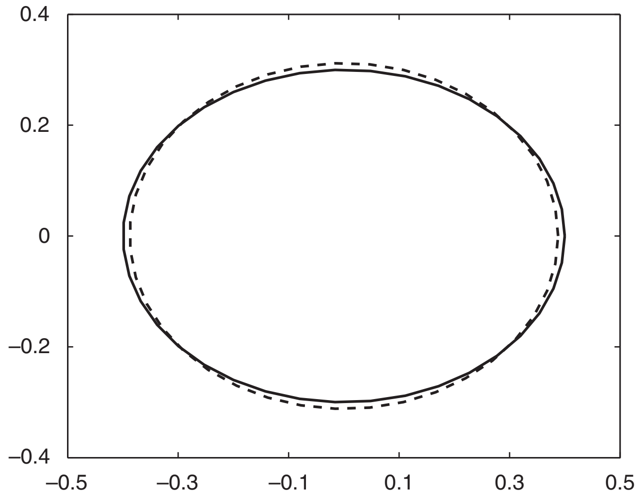

Figure 3. The numerical result of case 1 with $v = 0.01$.

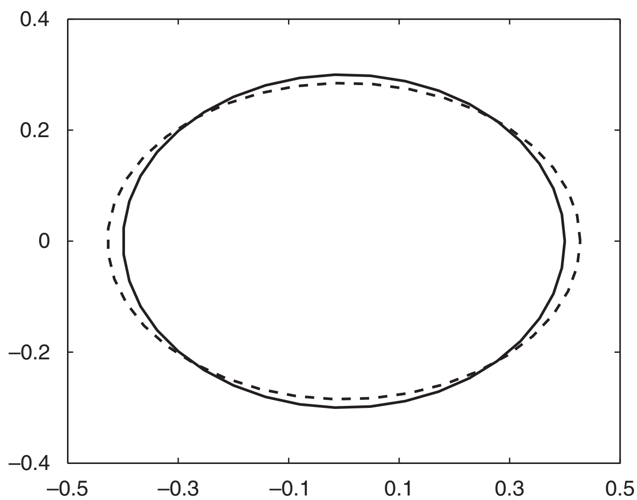

Figure 4. The numerical result of case 1 with $v = 0.005$.

Step (4): Apply the Gauss–Newton method,

$$\alpha^{n+1} = \alpha^n - (J(\alpha^n)^{\mathrm{T}} J(\alpha^n))^{-1} J(\alpha^n) r(\alpha^n)$$

where $r(\alpha^n) = (F_1(\alpha^n) - P_1, \ldots, F_Q(\alpha^n) - P_Q)^{\mathrm{T}}$. If

$$\sum_{i=1}^{Q} |F_i(\alpha^n) - P_i| + \mu \|\alpha\|^2 < \varepsilon$$

where μ is a regularization parameter, then terminate, otherwise go back to step (2).

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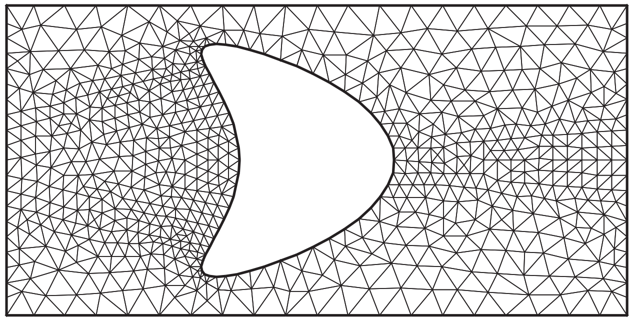

Figure 5. Initial mesh in case 2 with 448 nodes.

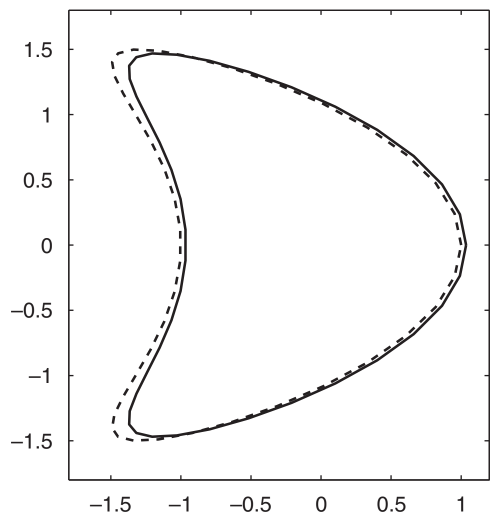

Figure 6. The numerical result of case 2 with $v = 0.01$.

In order to illustrate the feasibility of the proposed algorithm, we reconstruct several different interior boundaries:

Case 1: A parabolic: $x^2/16 + y^2/9 = \frac{1}{25}$.
Case 2: A kite-shaped curve given by the function $\{x(t) = \cos(t) + 0.65*(\cos(2t) - 1), y(t) = 1.5*\sin(t), t \in [0, 2\pi]\}$.
Case 3: A bean-shaped curve given in the following form, for any $t \in [0, 2\pi]$

$$x(t) = \sqrt{\cos^2(t) + 0.26*\sin(t + 0.5)}*\cos(t)$$

$$y(t) = \sqrt{\cos^2(t) + 0.26*\sin(t + 0.5)}*\sin(t)$$

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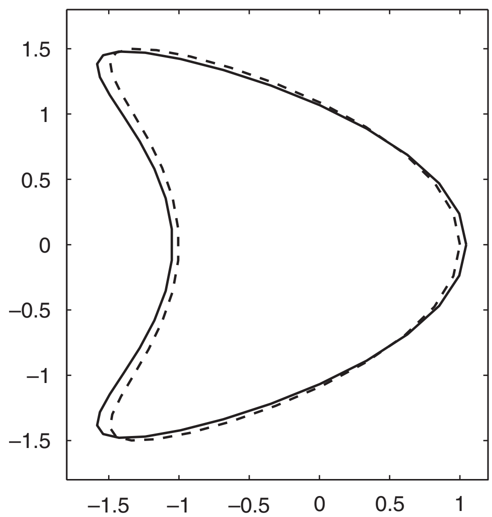

Figure 7. The numerical result of case 2 with $v = 0.005$.

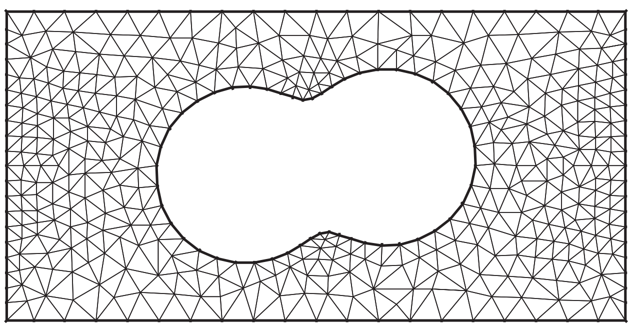

Figure 8. Initial mesh in case 3 with 286 nodes.

The boundary of the domain Ω has two parts: the exterior boundary Γ_1 which is fixed; the interior boundary Γ_2 which is to be reconstructed (see Figures 1, 5 and 8). We use the finite element method to simulate numerically. In the following figures, the solid line for the interior curve represents the exact boundary and the dashed line gives the approximate boundary.

In case 1, Figures 2–4 give the comparison between the target shape with reconstructed shape for the viscosity coefficient $v = 0.2, 0.01, 0.005$, respectively. We find that for $v = 0.2, 0.01$, we have a nice reconstruction, but for $v = 0.005$, the result is not satisfactory in Figure 4. In case 2, Figures 6 and 7 represent the comparison between the exact boundary with reconstructed boundary for the viscosity coefficient $v = 0.01$ and 0.005. Figures 9 and 10 indicate the numerical result of the shape reconstruction for the viscosity coefficient $v = 0.02, 0.01$ of case 3.

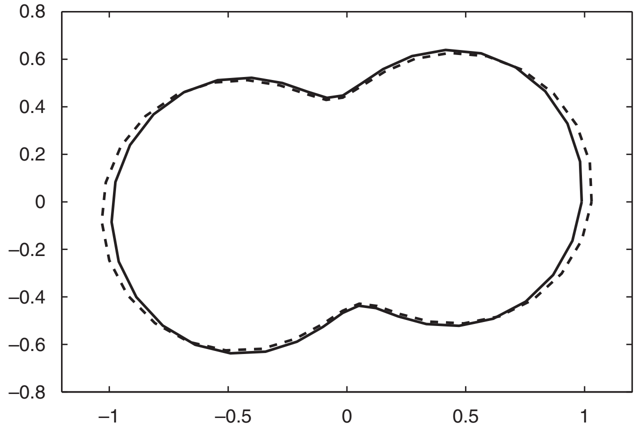

Figure 9. The numerical result of case 3 with $v=0.02$.

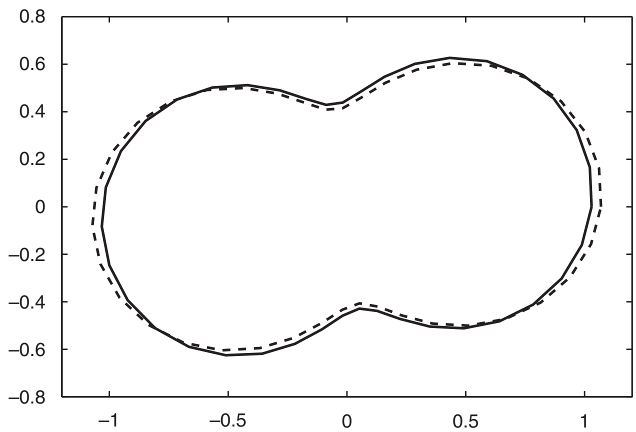

Figure 10. The numerical result of case 3 with $v=0.01$.

Finally, the numerical examples show the feasibility of the proposed algorithm and further research is necessary on efficient implementations.

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